Simplified vacuum energy expressions for radial backgrounds and domain walls

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Abstract
We extend our previous results of simplified expressions for functional determinants for radial Schrödinger operators to the computation of vacuum energy, or mass corrections, for static but spatially radial backgrounds, and for domain wall configurations. Our method is based on the zeta function approach to the Gel'fand–Yaglom theorem, suitably extended to higher-dimensional systems on separable manifolds. We find new expressions that are easy to implement numerically, for both zero and non-zero temperatures.

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1. Introduction

Determinants of differential operators occur naturally in many applications in mathematical and theoretical physics, and also have inherent mathematical interest since they encode certain spectral properties of differential operators. Physically, such determinants arise in semiclassical and one-loop approximations in quantum mechanics and quantum field theory [11, 15, 32, 34, 46, 47, 49]. Determinants of free Laplacians and free Dirac operators have been extensively studied [10, 12, 14, 20, 30, 38, 44, 48], but much less is known about operators involving an arbitrary potential function. For ordinary (i.e., one-dimensional) differential operators, a general theory has been developed for determinants of such operators [24, 25, 38–42]. In this paper we extend these results to a broad class of separable *partial* differential operators. Our approach is based on the zeta function definition of the determinant [30, 44], together with dimensional regularization and renormalization. Thus, it is similar in spirit to some previous approaches [2, 3, 7, 8, 17, 26–28, 33, 43, 45, 51, 54], but our main new result is that we are able to simplify the final expressions considerably by systematically separating out finite parts of the subtractions.

Let $x \in \mathbb{R}$ and $y \in M$, with M some $(d-1)$-dimensional manifold. We consider functional determinants of Laplace-type operators of the form

$$L = -\frac{d^2}{dx^2} - \Delta_y + m^2 + V(x) + W(y). \tag{1.1}$$

Using a separation of variables,

$$\phi(x, y) = X(x)Y(y),$$

the eigenvalues and eigenfunctions for this operator are determined by

$$\left(-\frac{d^2}{dx^2} + m^2 + \lambda^2 + V(x)\right) X(x) = k^2 X(x),$$

where

$$(-\Delta_y + W(y))Y(y) = \lambda^2 Y(y).$$

In this paper, we consider two situations in which we can obtain computationally simple expressions for the determinant of L.

(i) The first case is where $V(x) = 0$, and $W(y)$ defines a radial potential on the $(d-1)$-dimensional space M. The physical application motivating this case is the computation of induced vacuum energy, or one-loop mass corrections, in quantum field theory for backgrounds that are *static* (we regard x as the Euclidean time coordinate) and spatially radially symmetric (M is the spatial manifold). For a bosonic field the vacuum energy is given by

$$\Delta E = -\tfrac{1}{2}\mathrm{tr}_p \log \det(-\Delta_y + W(y) + m^2 + p^2), \tag{1.2}$$

where tr_p is a sum over Matsubara modes at finite temperature, or an integral at zero temperature [35]:

$$\mathrm{tr}_p g(p) \equiv \begin{cases} \dfrac{1}{\beta} \displaystyle\sum_{n=-\infty}^{\infty} g(p_n), \quad p_n = \dfrac{2\pi n}{\beta}, \quad \text{finite temperature,} \\[2ex] \displaystyle\int_{-\infty}^{\infty} \dfrac{dp}{2\pi} g(p), \quad \text{zero temperature.} \end{cases} \tag{1.3}$$

(ii) The second case is where $V(x)$ is non-zero, and M is a non-trivial $(d-1)$-dimensional manifold such as a torus T^{d-1}, or a sphere S^{d-1}, with $W(y) = 0$. In this case, we study the effect of the non-trivial topology of M on the determinant.

In each of these cases, there is a natural separation of variables, but the sum over the remaining eigenvalues (partial waves, Matsubara modes or Kaluza–Klein modes) is divergent and must be regularized and renormalized. We solve these regularization and renormalization issues using the zeta function approach.

2. Vacuum energy in a spherically symmetric background field

In this section, we find a computationally simple expression for the vacuum energy (1.2) for a bosonic field in a static but spatially radial background. We work at finite temperature, but the limit of the final result to zero temperature will be clear. A possible first guess would be to take the result of [16] for a radial determinant in dimension $(d-1)$, replace m^2 by $m^2 + p_n^2$, and trace over the eigenvalues p_n. However, while the finite renormalized (in $(d-1)$ dimensions) determinant for any given p_n can be computed using the results of [16], the subsequent trace over p_n is divergent. This divergence contains important physics and must

be treated appropriately, namely renormalizing in d dimensions is needed. Our main result is equation (2.24), which expresses the finite and renormalized log determinant in $(3 + 1)$-dimensions in a form that is easy to compute numerically.

We begin with the zeta function, and by standard techniques [8, 38], we express the zeta function in terms of the Jost function f_l as

$$\zeta(s) = \frac{\sin \pi s}{\pi \beta} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \deg(l; d-1) \int_{\sqrt{p_n^2+m^2}}^{\infty} dk \left(k^2 - p_n^2 - m^2\right)^{-s} \frac{d}{dk} \ln f_l(ik), \qquad (2.1)$$

where $\deg(l; d-1)$ is the degeneracy of the lth partial wave for the $(d-1)$-dimensional radial Laplacian:

$$\deg(l; d-1) = \frac{(2l+d-3)(l+d-4)!}{l!(d-3)!}. \qquad (2.2)$$

The task is to find the analytic continuation of the zeta function so that the zeta function and its derivative can be analyzed in the neighborhood of $s = 0$. Such an analytic continuation can be achieved by adding and subtracting the uniform asymptotic behavior, $f_l^{\text{asym}}(ik)$, of the Jost function $f_l(ik)$. This asymptotic behavior is well known from scattering theory [53]:

$$\ln f_l^{\text{asym}}(ik) \sim \frac{1}{2\nu} \int_0^{\infty} dr \frac{r W(r)}{\left(1 + \frac{k^2 r^2}{\nu^2}\right)^{1/2}}$$

$$+ \frac{1}{16\nu^3} \int_0^{\infty} dr \frac{r W(r)}{\left(1 + \frac{k^2 r^2}{\nu^2}\right)^{3/2}} \left[1 - \frac{6}{\left(1 + \frac{k^2 r^2}{\nu^2}\right)} + \frac{5}{\left(1 + \frac{k^2 r^2}{\nu^2}\right)^2}\right]$$

$$- \frac{1}{8\nu^3} \int_0^{\infty} dr \frac{r^3 W^2(r)}{\left(1 + \frac{k^2 r^2}{\nu^2}\right)^{3/2}} + \cdots, \qquad (2.3)$$

where for a $(d-1)$-dimensional radial system

$$\nu \equiv l + \frac{d-1}{2} - 1. \qquad (2.4)$$

Since we are interested in renormalizable theories with $d \leqslant 4$, it is sufficient to consider this many terms in the expansion[3]. Eventually, the convergence of our final expressions could be accelerated even more using further subtractions, as discussed in [31] for effective actions, but we do not pursue this here. For $d = 3$ we must first separate out the $l = 0$ term. We first concentrate on $d = 4$, and so $\nu \neq 0$.

Therefore, *defining* the function $\ln f_l^{\text{asym}}(ik)$ to be just the terms shown in (2.3), we are naturally led to the splitting:

$$\zeta(s) = \zeta_f(s) + \zeta_{as}(s), \qquad (2.5)$$

with the definitions

$$\zeta_f(s) = \frac{\sin \pi s}{\pi \beta} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \deg(l; d-1) \int_{\sqrt{p_n^2+m^2}}^{\infty} dk \left(k^2 - p_n^2 - m^2\right)^{-s}$$

$$\times \frac{d}{dk} \left[\ln f_l(ik) - \ln f_l^{\text{asym}}(ik)\right], \qquad (2.6)$$

$$\zeta_{as}(s) = \frac{\sin \pi s}{\pi \beta} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \deg(l; d-1) \int_{\sqrt{p_n^2+m^2}}^{\infty} dk \left(k^2 - p_n^2 - m^2\right)^{-s} \frac{d}{dk} \ln f_l^{\text{asym}}(ik). \qquad (2.7)$$

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[3] A natural mathematical generalization of a finite determinant in any dimension is possible [52], but requires higher order terms in this asymptotic expansion.

Then, by construction, $\zeta_f(s)$ is well defined about $s = 0$, and $\zeta_f'(0)$ is finite and given by

$$\zeta_f'(0) = -\frac{1}{\beta} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \deg(l; d-1) \left[\ln f_l\left(i\sqrt{p_n^2 + m^2}\right) - \ln f_l^{\mathrm{asym}}\left(i\sqrt{p_n^2 + m^2}\right) \right]. \tag{2.8}$$

Likewise, $\zeta_{as}(s)$ can be analytically continued to the neighborhood of $s = 0$, and its derivative $\zeta_{as}'(0)$ can be defined there.

Our main observation is that the final formula that results from this procedure, as reported in [7, 8], may be dramatically simplified owing to cancellations of finite terms between $\zeta_f'(0)$ and $\zeta_{as}'(0)$. These finite terms can be cancelled at the beginning, leading to much simpler expressions for each of $\zeta_f'(0)$ and $\zeta_{as}'(0)$, with their sum being unchanged. This leads to an expression that is precisely equal, but is significantly easier to compute numerically.

2.1. Rewriting $\zeta_f'(0)$

Our procedure for rewriting $\zeta_f'(0)$ is based on the simple observation that we can write

$$\left(1 + \frac{(p_n^2 + m^2)r^2}{\nu^2}\right) = \left(1 + \frac{p_n^2 r^2}{\nu^2}\right)\left(1 + \frac{m^2 r^2}{\nu^2 + p_n^2 r^2}\right) \tag{2.9}$$

and the second factor can now be expanded at both large partial wave number (i.e., large l, or large ν) and large p_n. Performing a large-l expansion directly on the left-hand side is not possible because $(p_n^2 + m^2)r^2/\nu^2$ is not a small parameter, as p_n can become arbitrarily large.

Proceeding in this fashion we can separate $\ln f_l^{\mathrm{asym}}\left(i\sqrt{p_n^2 + m^2}\right)$ according to the order of its terms at large $(\nu^2 + p_n^2 r^2)$:

$$\ln f_l^{\mathrm{asym}}\left(i\sqrt{p_n^2 + m^2}\right) \equiv \ln f_l^{\mathrm{asym},(1)}\left(i\sqrt{p_n^2 + m^2}\right) + \ln f_l^{\mathrm{asym},(2)}\left(i\sqrt{p_n^2 + m^2}\right), \tag{2.10}$$

where

$$\ln f_l^{\mathrm{asym},(1)}\left(i\sqrt{p_n^2 + m^2}\right) = \frac{1}{2\nu} \int_0^\infty dr \frac{r W(r)}{\left(1 + \frac{p_n^2 r^2}{\nu^2}\right)^{1/2}} - \frac{1}{8\nu^3} \int_0^\infty dr \frac{r^3 W(r)(W(r) + 2m^2)}{\left(1 + \frac{p_n^2 r^2}{\nu^2}\right)^{3/2}}$$

$$+ \frac{1}{16\nu^3} \int_0^\infty dr \frac{r W(r)}{\left(1 + \frac{p_n^2 r^2}{\nu^2}\right)^{3/2}} \left[1 - \frac{6}{\left(1 + \frac{p_n^2 r^2}{\nu^2}\right)} + \frac{5}{\left(1 + \frac{p_n^2 r^2}{\nu^2}\right)^2}\right] \tag{2.11}$$

contains terms of $\mathcal{O}\left((\nu^2 + p_n^2 r^2)^{-1/2}\right)$ and $\mathcal{O}\left((\nu^2 + p_n^2 r^2)^{-3/2}\right)$, while

$$\ln f_l^{\mathrm{asym},(2)}\left(i\sqrt{p_n^2 + m^2}\right) = \frac{1}{2\nu} \int_0^\infty dr \frac{r W(r)}{\left(1 + \frac{p_n^2 r^2}{\nu^2}\right)^{1/2}} \left[\frac{1}{\left(1 + \frac{m^2 r^2}{\nu^2 + p_n^2 r^2}\right)^{1/2}} - 1 + \frac{1}{2}\frac{m^2 r^2}{\nu^2 + p_n^2 r^2}\right]$$

$$- \frac{1}{8\nu^3} \int_0^\infty dr \frac{r^3 W^2(r)}{\left(1 + \frac{p_n^2 r^2}{\nu^2}\right)^{3/2}} \left[\frac{1}{\left(1 + \frac{m^2 r^2}{\nu^2 + p_n^2 r^2}\right)^{3/2}} - 1\right]$$

$$+ \frac{1}{16\nu^3} \int_0^\infty dr \frac{r W(r)}{\left(1 + \frac{p_n^2 r^2}{\nu^2}\right)^{3/2}} \left[\frac{1}{\left(1 + \frac{m^2 r^2}{\nu^2 + p_n^2 r^2}\right)^{3/2}} - 1\right]$$

$$- \frac{3}{8\nu^3} \int_0^\infty dr \frac{r W(r)}{\left(1 + \frac{p_n^2 r^2}{\nu^2}\right)^{5/2}} \left[\frac{1}{\left(1 + \frac{m^2 r^2}{\nu^2 + p_n^2 r^2}\right)^{5/2}} - 1\right]$$

$$+ \frac{5}{16\nu^3} \int_0^\infty dr \frac{r W(r)}{\left(1 + \frac{p_n^2 r^2}{\nu^2}\right)^{7/2}} \left[\frac{1}{\left(1 + \frac{m^2 r^2}{\nu^2 + p_n^2 r^2}\right)^{7/2}} - 1\right] \tag{2.12}$$

contains terms of order $\mathcal{O}\big((v^2 + p_n^2 r^2)^{-5/2}\big)$. Thus, for $\ln f_l^{\mathrm{asym},(2)}\big(\mathrm{i}\sqrt{p_n^2 + m^2}\big)$, the l- and n-summations in (2.8) are finite. It is therefore sufficient to subtract only $\ln f_l^{\mathrm{asym},(1)}\big(\mathrm{i}\sqrt{p_n^2 + m^2}\big)$ from $\ln f_l\big(\mathrm{i}\sqrt{p_n^2 + m^2}\big)$ to obtain a finite answer. Thus, we rewrite (2.8) as

$$
\zeta_f'(0) = -\frac{1}{\beta} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \deg(l; d-1)\Big[\ln f_l\big(\mathrm{i}\sqrt{p_n^2 + m^2}\big) - \ln f_l^{\mathrm{asym},(1)}\big(\mathrm{i}\sqrt{p_n^2 + m^2}\big)\Big]
$$
$$
+ \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \deg(l; d-1)\Big[\ln f_l^{\mathrm{asym},(2)}\big(\mathrm{i}\sqrt{p_n^2 + m^2}\big)\Big]. \tag{2.13}
$$

By construction, the summations are *separately finite*, and the total is precisely equal to the expression in (2.8). The main rationale behind this splitting is that we show below that the entire contribution of the sum over $\ln f_l^{\mathrm{asym},(2)}\big(\mathrm{i}\sqrt{p_n^2 + m^2}\big)$ in (2.13), which is completely finite, is cancelled precisely by identical terms coming from $\zeta_{as}'(0)$. It is therefore computationally redundant to compute these terms.

2.2. Rewriting $\zeta_{as}'(0)$

A similar strategy is applied to the evaluation of $\zeta_{as}'(0)$. We begin with the definition (2.7) of the zeta function $\zeta_{as}(s)$, and perform the k integrations using the basic dimensional regularization formula

$$
\int_{\sqrt{p_n^2 + m^2}}^{\infty} \mathrm{d}k\,(k^2 - p_n^2 - m^2)^{-s} \frac{\mathrm{d}}{\mathrm{d}k}\left(1 + \frac{k^2 r^2}{v^2}\right)^{-\frac{N}{2}} = -\frac{\pi\,\Gamma\big(s + \frac{N}{2}\big)}{\sin(\pi s)\Gamma\big(\frac{N}{2}\big)\Gamma(s)}
$$
$$
\times \frac{r^{2s} v^{-2s}}{\left(1 + \frac{(p_n^2 + m^2)r^2}{v^2}\right)^{s+N/2}}. \tag{2.14}
$$

Referring to (2.3), we see that each term in $\zeta_{as}(s)$ is of the form where this integration formula can be applied. Therefore, the k integrals in (2.7) produce a set of terms of the form

$$
\left(1 + \frac{(p_n^2 + m^2)r^2}{v^2}\right)^{s+N/2} \tag{2.15}
$$

for $N = 1, 3, 5, 7$. We now make an expansion based on the decomposition in (2.9), which naturally separates $\zeta_{as}(s)$ into two parts, mirroring the separation of $\ln f_l^{\mathrm{asym}}\big(\mathrm{i}\sqrt{p_n^2 + m^2}\big)$ in (2.10):

$$
\zeta_{as}(s) \equiv \zeta_{as}^{(1)}(s) + \zeta_{as}^{(2)}(s), \tag{2.16}
$$

for details see appendix A. The first term, $\zeta_{as}^{(1)}(s)$, is

$$
\zeta_{as}^{(1)}(s) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \deg(l; d-1) \left\{ -\frac{1}{2} \frac{\Gamma\big(s + \frac{1}{2}\big)}{\Gamma\big(\frac{1}{2}\big)\Gamma(s)} \int_0^{\infty} \mathrm{d}r\, \frac{\frac{r^{2s+1}}{v^{2s+1}} W(r)}{\big(1 + \frac{p_n^2 r^2}{v^2}\big)^{s+1/2}} \right.
$$
$$
+ \frac{1}{8} \frac{\Gamma\big(s + \frac{3}{2}\big)}{\Gamma\big(\frac{3}{2}\big)\Gamma(s)} \int_0^{\infty} \mathrm{d}r\, \frac{\frac{r^{2s+3}}{v^{2s+3}} W(r)(W(r) + 2m^2)}{\big(1 + \frac{p_n^2 r^2}{v^2}\big)^{s+3/2}} - \frac{1}{16v^2} \frac{\Gamma\big(s + \frac{3}{2}\big)}{\Gamma\big(\frac{3}{2}\big)\Gamma(s)}
$$
$$
\left. \times \int_0^{\infty} \mathrm{d}r\, \frac{\frac{r^{2s+1}}{v^{2s+1}} W(r)}{\big(1 + \frac{p_n^2 r^2}{v^2}\big)^{s+3/2}} \left[1 - \frac{4\big(s + \frac{3}{2}\big)}{\big(1 + \frac{p_n^2 r^2}{v^2}\big)} + \frac{\frac{4}{3}\big(s + \frac{3}{2}\big)\big(s + \frac{5}{2}\big)}{\big(1 + \frac{p_n^2 r^2}{v^2}\big)^2}\right] \right\}. \tag{2.17}
$$

The second term in the decomposition (2.16) is

$$
\zeta_{as}^{(2)}(s) = \frac{1}{\beta} \frac{\Gamma\left(s+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(s)} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \deg(l; d-1) \left\{ -\frac{1}{2} \int_0^{\infty} dr \frac{\frac{r^{2s+1}}{\nu^{2s+1}} W(r)}{\left(1 + \frac{p_n^2 r^2}{\nu^2}\right)^{s+1/2}} \right.
$$

$$
\times \left[\frac{1}{\left(1 + \frac{m^2 r^2}{\nu^2 + p_n^2 r^2}\right)^{s+1/2}} - 1 + \left(s + \frac{1}{2}\right) \frac{m^2 r^2}{\nu^2 + p_n^2 r^2} \right]
$$

$$
+ \frac{\left(s + \frac{1}{2}\right)}{4} \int_0^{\infty} dr \frac{\frac{r^{2s+3}}{\nu^{2s+3}} W^2(r)}{\left(1 + \frac{p_n^2 r^2}{\nu^2}\right)^{s+3/2}} \left[\frac{1}{\left(1 + \frac{m^2 r^2}{\nu^2 + p_n^2 r^2}\right)^{s+3/2}} - 1 \right]
$$

$$
- \frac{\left(s + \frac{1}{2}\right)}{8\nu^2} \int_0^{\infty} dr \frac{\frac{r^{2s+1}}{\nu^{2s+1}} W(r)}{\left(1 + \frac{p_n^2 r^2}{\nu^2}\right)^{s+3/2}} \left[\frac{1}{\left(1 + \frac{m^2 r^2}{\nu^2 + p_n^2 r^2}\right)^{s+3/2}} - 1 \right]
$$

$$
+ \frac{\left(s + \frac{1}{2}\right)\left(s + \frac{3}{2}\right)}{2\nu^2} \int_0^{\infty} dr \frac{\frac{r^{2s+1}}{\nu^{2s+1}} W(r)}{\left(1 + \frac{p_n^2 r^2}{\nu^2}\right)^{s+5/2}} \left[\frac{1}{\left(1 + \frac{m^2 r^2}{\nu^2 + p_n^2 r^2}\right)^{s+5/2}} - 1 \right]
$$

$$
\left. - \frac{\left(s + \frac{1}{2}\right)\left(s + \frac{3}{2}\right)\left(s + \frac{5}{2}\right)}{6\nu^2} \int_0^{\infty} dr \frac{\frac{r^{2s+1}}{\nu^{2s+1}} W(r)}{\left(1 + \frac{p_n^2 r^2}{\nu^2}\right)^{s+7/2}} \left[\frac{1}{\left(1 + \frac{m^2 r^2}{\nu^2 + p_n^2 r^2}\right)^{s+7/2}} - 1 \right] \right\}.
$$

$$(2.18)$$

It is clear that, by construction, $\zeta_{as}^{(2)}(s)$ is regular at $s = 0$, and a simple computation yields

$$
\left[\frac{d}{ds} \zeta_{as}^{(2)}(s) \right]_{s=0} = -\frac{1}{\beta} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \deg(l; d-1) \left[\ln f_l^{\text{asym},(2)} \left(i\sqrt{p_n^2 + m^2} \right) \right] \tag{2.19}
$$

so that it cancels exactly the second, finite, sum in (2.13) for $\zeta_f'(0)$. Thus, we only need to compute $\left[\frac{d}{ds} \zeta_{as}^{(1)}(s) \right]_{s=0}$.

To compute $\left[\frac{d}{ds} \zeta_{as}^{(1)}(s) \right]_{s=0}$, it is useful to define the Epstein-like zeta function

$$
E_d^{(k)}(s, a) \equiv \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \deg(l; d-1) \frac{\nu^k}{(\nu^2 + a^2 n^2)^s}. \tag{2.20}
$$

Then

$$
\zeta_{as}^{(1)}(s) = -\frac{\Gamma\left(s + \frac{1}{2}\right)}{2\sqrt{\pi}\,\Gamma(s)} \int_0^{\infty} dr\, W r^{1+2s} E_d^{(0)}\left(s + \frac{1}{2}, \frac{2\pi r}{\beta}\right)
$$

$$
+ \frac{\Gamma\left(s + \frac{3}{2}\right)}{4\sqrt{\pi}\,\Gamma(s)} \int_0^{\infty} dr\, W(W + 2m^2) r^{2s+3} E_d^{(0)}\left(s + \frac{3}{2}, \frac{2\pi r}{\beta}\right)
$$

$$
- \frac{\Gamma\left(s + \frac{3}{2}\right)}{8\sqrt{\pi}\,\Gamma(s)} \int_0^{\infty} dr\, W r^{1+2s} \left[E_d^{(0)}\left(s + \frac{3}{2}, \frac{2\pi r}{\beta}\right) - 4\left(s + \frac{3}{2}\right) \right.
$$

$$
\left. \times E_d^{(2)}\left(s + \frac{5}{2}, \frac{2\pi r}{\beta}\right) + \frac{4}{3}\left(s + \frac{3}{2}\right)\left(s + \frac{5}{2}\right) E_d^{(4)}\left(s + \frac{7}{2}, \frac{2\pi r}{\beta}\right) \right]. \tag{2.21}
$$

To compute $\left[\frac{d}{ds} \zeta_{as}^{(1)}(s) \right]_{s=0}$, we need the behavior of the zeta functions $E_d^{(k)}(s, z)$ in the vicinity of $s = 1/2, 3/2, 5/2, 7/2$. This behavior is particularly simple in the zero temperature limit, in which we can replace the sum over Matsubara modes by an integral as in (1.3) [35]. Then

the zeta functions in (2.21) reduce to Hurwitz zeta functions, see (A.3), (A.5) and (A.6). For example, in $d = 4$, where $\deg(l; d - 1) = 2l + 1 \equiv 2\nu$, we find at $T = 0$

$$
\begin{aligned}
\zeta_{as}^{(1)}(s) = {}& -\frac{1}{2\pi} \int_0^\infty dr \, W r^{2s} (2^{2s-1} - 1) \zeta_R(2s - 1) \\
& + \frac{1}{4\pi} \int_0^\infty dr \, W(W + 2m^2) r^{2s+2} s (2^{2s+1} - 1) \zeta_R(2s + 1) \\
& - \frac{1}{8\pi} \int_0^\infty dr \, W r^{2s} \left[s - 4s(s + 1) + \frac{4}{3} s(s + 1)(s + 2) \right] (2^{2s+1} - 1) \zeta_R(2s + 1).
\end{aligned}
$$
(2.22)

We thus find the following simple expression for the derivative at $s = 0$:

$$
\begin{aligned}
\left[\frac{d}{ds} \zeta_{as}^{(1)}(s) \right]_{s=0} = {}& \frac{1}{24\pi} [\gamma + 3 \ln 2 + 12 \zeta_R'(-1)] \int_0^\infty dr \, W(r) \\
& + \frac{1}{4\pi} \int_0^\infty dr \, W(r)(W(r) + 2m^2) r^2 [\gamma + \ln(4r)].
\end{aligned}
$$
(2.23)

Note that this expression is very simple—it does not involve any summations.

2.3. Combining the results for $\zeta_f'(0)$ and $\zeta_{as}'(0)$

In the previous two subsections we showed that each of $\zeta_f'(0)$ and $\zeta_{as}'(0)$ could be split naturally into two parts, with all parts being manifestly finite, and in such a way that the second part of $\zeta_f'(0)$ in (2.13) cancels against the second part of $\zeta_{as}'(0)$ in (2.19). Thus, we are left with a much simpler expression for the net $\zeta'(0)$:

$$
\begin{aligned}
[\zeta'(0)]_{d=4} = {}& -\frac{1}{2\pi} \int_{-\infty}^\infty dp \sum_{l=0}^\infty (2\nu) \left[\ln f_l(i\sqrt{p^2 + m^2}) - \frac{1}{2} \int_0^\infty dr \frac{r W}{(\nu^2 + p^2 r^2)^{1/2}} \right. \\
& + \frac{1}{8} \int_0^\infty dr \frac{r^3 W(W + 2m^2)}{(\nu^2 + p^2 r^2)^{3/2}} - \frac{1}{16} \int_0^\infty \frac{dr \, r W(r)}{(\nu^2 + p^2 r^2)^{3/2}} \\
& \left. \times \left(1 - \frac{6\nu^2}{(\nu^2 + p^2 r^2)} + \frac{5\nu^4}{(\nu^2 + p^2 r^2)^2} \right) \right] \\
& + \frac{1}{24\pi} [\gamma + 3 \ln 2 + 12 \zeta_R'(-1)] \int_0^\infty dr \, W \\
& + \frac{1}{4\pi} \int_0^\infty dr \, W(W + 2m^2) r^2 [\gamma + \ln(4rm)]
\end{aligned}
$$
(2.24)

where $\nu = l + 1/2$ for $d = 4$; the vacuum energy now is $\Delta E = (1/2)\zeta'(0)$.

The result for finite temperature follows by the replacement

$$
\frac{1}{2\pi} \int_{-\infty}^\infty dp \rightarrow \frac{1}{\beta} \sum_{-\infty}^\infty \qquad \text{and} \qquad p \rightarrow p_n.
$$

Furthermore, there are exponentially damped contributions as $\beta \rightarrow \infty$, involving series over Bessel functions as described in appendix A.

Equation (2.24) and its finite temperature version are our main results in this section. The computational simplicity of this result in comparison to previous expressions is worth stressing. For each Matsubara mode p_n, and each partial wave l, the logarithm of the Jost function, $\ln f_l(i\sqrt{p_n^2 + m^2})$, can be evaluated simply and efficiently using the radial version of the Gel'fand–Yaglom theorem, as described in [15]. Our result states that, with the subtractions

in (2.24), the double sum over n and l converges. These subtraction terms are simple integrals, easy to evaluate numerically. The finite counterterm on the last line does not involve any summation, and can be evaluated once and for all given the radial potential $W(r)$. By contrast, the corresponding subtraction terms, and also the counterterms are significantly more involved in [8]. The specific form of the finite term on the last line of (2.24) arises through our use of dimensional regularization, and we have renormalized on-shell at renormalization scale $\mu = m$. It is well known how to convert from a given renormalization scheme to another by finite counterterms [4], so our expression (2.24) gives not just the finite determinant, but the finite and renormalized vacuum energy.

In $d = 3$ the $l = 0$ mode needs separate treatment, the $l \geqslant 1$ modes are dealt with as before; see appendix A for details. Again, at $T = 0$ the answer is surprisingly simple and reads

$$\zeta'(0) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}p \left[\ln f_0(\mathrm{i}\sqrt{p^2 + m^2}) - \frac{1}{2\sqrt{p^2 + m^2}} \int_0^\infty \mathrm{d}r \, W(r) \right]$$
$$- \frac{1}{\pi} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}p \left[\ln f_l(\mathrm{i}\sqrt{p^2 + m^2}) - \frac{1}{2} \int_0^\infty \mathrm{d}r \frac{r \, W(r)}{(l^2 + p^2 r^2)^{\frac{1}{2}}} \right]$$
$$+ \frac{1}{2\pi} \int_0^\infty \mathrm{d}r \, W(r) \ln(2\pi r m).$$

The case $d = 2$ follows from (3.6) once we put $L = \beta$. For finite temperature the remarks below (2.24) remain valid.

3. Compactification with non-trivial $V(x)$

We now turn to the opposite limit, in which the non-trivial potential is not a radial potential on the $(d-1)$-dimensional manifold M, but a one-dimensional potential for $x \in \mathbb{R}$. Physically, this corresponds to a domain wall configuration in the x-direction, with M describing the transverse directions. So let us assume $V(x)$ is the non-trivial potential with suitable asymptotic properties as defined below in terms of the Jost function. For the zeta function analysis we follow the one-dimensional scattering approach as described in [5]. In this calculation we assume $\lambda^2 > 0$; the contribution for $\lambda^2 = 0$ is easily added at the end. Starting off as usual [38] with a suitable contour in the complex plane, the zeta function for the above operator, after shifting it to the imaginary axis, reads

$$\zeta(s) = -\frac{\sin(\pi s)}{\pi} \sum_\lambda \int_{\sqrt{m^2+\lambda^2}}^{\infty} \mathrm{d}p (p^2 - m^2 - \lambda^2)^{-s} \frac{\partial}{\partial p} \ln s_{11}(\mathrm{i}p) \qquad (3.1)$$

with s_{11} a suitable element of the S-matrix. The asymptotics of $s_{11}(\mathrm{i}p)$ for large p is known [55] and reads

$$\ln s_{11}(\mathrm{i}p) = -\frac{1}{2p} \int_{-\infty}^{\infty} \mathrm{d}x \, V(x) + \frac{1}{8p^3} \int_{-\infty}^{\infty} \mathrm{d}x (V(x))^2 + \mathcal{O}(p^{-5}). \qquad (3.2)$$

Applying the usual procedure of subtracting and adding the asymptotic behavior, we rewrite the zeta function as

$$\zeta(s) = -\frac{\sin(\pi s)}{\pi} \sum_\lambda \int_{\sqrt{m^2+\lambda^2}}^{\infty} \mathrm{d}p (p^2 - m^2 - \lambda^2)^{-s} \frac{\partial}{\partial p} \left\{ \ln s_{11}(\mathrm{i}p) + \frac{1}{2p} \int_{-\infty}^{\infty} \mathrm{d}x \, V(x) \right\}$$
$$+ \frac{\sin(\pi s)}{\pi} \sum_\lambda \int_{\sqrt{m^2+\lambda^2}}^{\infty} \mathrm{d}p (p^2 - m^2 - \lambda^2)^{-s} \frac{\partial}{\partial p} \left(\frac{1}{2p} \int_{-\infty}^{\infty} \mathrm{d}x \, V(x) \right). \qquad (3.3)$$

Here we have only subtracted the first term in the asymptotic expansion, which turns out to be sufficient in $d = 2$ and $d = 3$. Subtracting more terms might be numerically helpful in these dimensions, and furthermore it is necessary in $d \geqslant 4$. The $d = 4$ result is given in section 3.3. But for ease of presentation we first focus on $d \leqslant 3$; the generalization is straightforward.

The above representation of the zeta function shows that

$$\zeta(s) = \zeta_f(s) + \zeta_{as}(s),$$

with

$$\zeta_f(s) = -\frac{\sin(\pi s)}{\pi} \sum_\lambda \int_{\sqrt{m^2+\lambda^2}}^\infty dp (p^2 - m^2 - \lambda^2)^{-s} \frac{\partial}{\partial p} \left\{ \ln s_{11}(ip) + \frac{1}{2p} \int_{-\infty}^\infty dx\, V(x) \right\}$$

and

$$\zeta_{as}(s) = -\frac{\sin \pi s}{2\pi} \int_{-\infty}^\infty dx\, V(x) \sum_\lambda \int_{\sqrt{m^2+\lambda^2}}^\infty dp \frac{(p^2 - m^2 - \lambda^2)^{-s}}{p^2}.$$

By construction, $\zeta_f(s)$ is well behaved about $s = 0$ and $\zeta_f'(0)$ is trivially calculated,

$$\zeta_f'(0) = \sum_\lambda \left\{ \ln s_{11}(i\sqrt{m^2 + \lambda^2}) + \frac{1}{2\sqrt{m^2 + \lambda^2}} \int_{-\infty}^\infty dx\, V(x) \right\}.$$

Cancellations between $\zeta_f'(0)$ and $\zeta_{as}'(0)$ are expected to occur if we expand the above expression further for $|\lambda| \gg 1$ because this has been observed in [16] for spherically symmetric potentials. In the range $\lambda \gg 1$ we use

$$(m^2 + \lambda^2)^{-1/2} = \frac{1}{|\lambda|}(1 + \mathcal{O}(|\lambda|^{-3})).$$

This allows us to rewrite the answer for $\zeta_f'(0)$ in the form

$$\zeta_f'(0) = \sum_\lambda \left\{ \ln s_{11}(i\sqrt{m^2 + \lambda^2}) + \frac{1}{2|\lambda|} \int_{-\infty}^\infty dx\, V(x) \right\}$$
$$+ \frac{1}{2} \sum_\lambda \left\{ \frac{1}{\sqrt{m^2 + \lambda^2}} - \frac{1}{|\lambda|} \right\} \int_{-\infty}^\infty dx\, V(x).$$

For $\zeta_{as}(s)$ we first perform the p-integration to obtain

$$\zeta_{as} = -\frac{\sin \pi s}{2\pi} \frac{\Gamma(1-s)\Gamma\left(\frac{1}{2}+s\right)}{\sqrt{\pi}} \int_{-\infty}^\infty dx\, V(x) \sum_\lambda (m^2 + \lambda^2)^{-s-1/2}.$$

This makes the introduction of the zeta function of the Laplace-type operator in the y-coordinate necessary, and we define

$$\zeta_y^m(s) = \sum_\lambda (m^2 + \lambda^2)^{-s}.$$

Therefore,

$$\zeta_{as}(s) = -\frac{\sin \pi s}{2\pi} \frac{\Gamma(1-s)\Gamma\left(\frac{1}{2}+s\right)}{\sqrt{\pi}} \zeta_y^m \left(s + \frac{1}{2} \right) \int_{-\infty}^\infty dx\, V(x)$$
$$= -\frac{\Gamma\left(s + \frac{1}{2}\right)}{2\sqrt{\pi}\Gamma(s)} \zeta_y^m \left(s + \frac{1}{2} \right) \int_{-\infty}^\infty dx\, V(x).$$

Applying a similar procedure as in $\zeta_f(s)$, we write

$$\zeta_y^m \left(s + \frac{1}{2} \right) = \zeta_y^0 \left(s + \frac{1}{2} \right) + \sum_\lambda \{(m^2 + \lambda^2)^{-s-1/2} - |\lambda|^{-2s-1}\}.$$

Using this splitting, we continue

$$\zeta'_{as}(0) = -\frac{1}{2} \int_{-\infty}^{\infty} dx\, V(x) \sum_{\lambda} \{(m^2 + \lambda^2)^{-1/2} - |\lambda|^{-1}\}$$

$$- \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} dx\, V(x) \frac{d}{ds}\bigg|_{s=0} \left(\frac{\Gamma\left(s + \frac{1}{2}\right)}{\Gamma(s)} \zeta_y^0 \left(s + \frac{1}{2}\right) \right).$$

In order to provide an answer as explicitly as possible, which means an answer that will be applicable to examples as easily as possible, we will assume that the structure of the zeta function on M is the standard one, i.e. for $s \approx 0$ we assume [50]

$$\zeta_y^0 \left(s + \frac{1}{2}\right) = \frac{1}{s} \operatorname{Res} \zeta_y^0 \left(\frac{1}{2}\right) + PP\zeta_y^0 \left(\frac{1}{2}\right) + \mathcal{O}(s).$$

This structure will apply for example for a smooth potential $W(y)$ on a compact manifold M, or for non-compact manifolds if the potential $W(y)$ is falling off sufficiently fast at infinity. With this structure assumed, it is easy to show that

$$\zeta'_{as}(0) = -\frac{1}{2} \int_{-\infty}^{\infty} dx\, V(x) \left\{ \sum_{\lambda} [(m^2 + \lambda^2)^{-1/2} - |\lambda|^{-1}] + PP\zeta_y^0 \left(\frac{1}{2}\right) - 2\ln 2 \operatorname{Res} \zeta_y^0 \left(\frac{1}{2}\right) \right\}.$$

Adding up the contributions from $\zeta'_f(0)$ and $\zeta'_{as}(0)$ we find

$$\zeta'(0) = \sum_{\lambda} \left\{ \ln s_{11}(i\sqrt{m^2 + \lambda^2}) + \frac{1}{2|\lambda|} \int_{-\infty}^{\infty} dx\, V(x) \right\}$$

$$+ \left\{ -\frac{1}{2} PP\zeta_y^0 \left(\frac{1}{2}\right) + \ln 2 \operatorname{Res} \zeta_y^0 \left(\frac{1}{2}\right) \right\} \int_{-\infty}^{\infty} dx\, V(x). \tag{3.4}$$

This seems to be the most compact answer in $d = 2$ and $d = 3$ one can find. If there are d_0 eigenvalues $\lambda = 0$, one needs to add the contribution $d_0 \ln s_{11}(im)$ to the above answer, with the understanding that the summation over λ in the subsequent quantities always omits the eigenvalue $\lambda = 0$.

Once the manifold M is specified to be a particular manifold, thereby defining the zeta functions $\zeta_y(s)$ and $\zeta_y^0(s)$, the above results can be made completely explicit. This is best seen for cases like the torus and the sphere with $W(y) = 0$ where final answers are given in terms of well-known special functions. In principle, it could also be done for the example of a ball, but the associated zeta functions are not readily expressed in terms of known functions, see [6, 9], and therefore we do not present details.

3.1. Example of the torus

d = 2: Let us assume one toroidally compactified dimension of length L. Then $\lambda_n^2 = (2\pi n/L)^2$, $n \in \mathbb{Z}$, so $d_0 = 1$, and

$$\zeta_y^0(s) = \sum_{n \in \mathbb{Z}/\{0\}} \left(\frac{2\pi n}{L}\right)^{-2s} = 2\left(\frac{2\pi}{L}\right)^{-2s} \zeta_R(2s). \tag{3.5}$$

In this case,

$$\operatorname{Res} \zeta_y^0 \left(\frac{1}{2}\right) = \frac{L}{2\pi}, \qquad PP\zeta_y^0 \left(\frac{1}{2}\right) = \frac{L}{\pi}\left(\gamma - \ln \frac{2\pi}{L}\right).$$

Therefore, the final answer with one toroidal dimension reads

$$\zeta'(0) = \ln s_{11}(im) + 2 \sum_{n=1}^{\infty} \left\{ \ln s_{11} \left(i \left[m^2 + \left(\frac{2\pi n}{L} \right)^2 \right]^{1/2} \right) + \frac{L}{4\pi n} \int_{-\infty}^{\infty} dx \, V(x) \right\}$$
$$- \frac{L}{2\pi} \left(\gamma + \ln \frac{L}{4\pi} \right) \int_{-\infty}^{\infty} dx \, V(x). \tag{3.6}$$

d = 3: Let us now assume two toroidally compactified dimensions of lengths L_1 and L_2. In this case the zeta function associated with the y-differential operator is an Epstein zeta function [21, 22],

$$\zeta_y^0(s) = (2\pi)^{-2s} \sum_{(n,j) \in \mathbb{Z}^2 / \{0\}} \left[\left(\frac{n}{L_1} \right)^2 + \left(\frac{j}{L_2} \right)^2 \right]^{-s}.$$

These functions, often defined as

$$Z_2(s; w_1, w_2) = \sum_{(n,j) \in \mathbb{Z}^2 / \{0\}} [w_1 n^2 + w_2 j^2]^{-s},$$

have well-understood analytical continuations [1, 18, 37]. Particularly suitable if one of the compactification lengths is sent to zero is

$$Z_2(s; w_1, w_2) = \frac{2}{w_2^s} \zeta_R(2s) + \frac{2\sqrt{\pi}}{\sqrt{w_2}} w_1^{\frac{1}{2}-s} \zeta_R(2s-1) \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}$$
$$+ \frac{8\pi^s}{\Gamma(s)\sqrt{w_2}} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \left[\sqrt{w_1 w_2} \frac{j}{n} \right]^{\frac{1}{2}-s} K_{\frac{1}{2}-s} \left(2\pi n j \sqrt{\frac{w_1}{w_2}} \right). \tag{3.7}$$

From this analytical continuation one easily derives that $\text{Res} \, \zeta_y^0(1/2) = 0$ (the singular contributions from the first two terms cancel), and

$$\zeta_y^0 \left(\frac{1}{2} \right) = \frac{L_2}{\pi} \left(\gamma + \ln \left[\frac{1}{4\pi} \frac{L_2}{L_1} \right] \right) + \frac{4L_2}{\pi} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} K_0 \left(2\pi n j \frac{L_2}{L_1} \right).$$

Using this in (3.4), we find

$$\zeta'(0) = \ln s_{11}(im) - \left\{ \frac{L_2}{2\pi} \left(\gamma + \ln \left[\frac{1}{4\pi} \frac{L_2}{L_1} \right] \right) + \frac{2L_2}{\pi} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} K_0 \left(2\pi n j \frac{L_2}{L_1} \right) \right\} \int_{-\infty}^{\infty} dx \, V(x)$$
$$+ \sum_{(n,j) \in \mathbb{Z}^2 / \{0\}} \left\{ \ln s_{11} \left(i \sqrt{m^2 + \left(\frac{2\pi n}{L_1} \right)^2 + \left(\frac{2\pi j}{L_2} \right)^2} \right) \right.$$
$$\left. + \frac{1}{4\pi} \frac{1}{\sqrt{\left(\frac{n}{L_1} \right)^2 + \left(\frac{j}{L_2} \right)^2}} \int_{-\infty}^{\infty} dx \, V(x) \right\}. \tag{3.8}$$

In case we consider toroidal compactification of equal sides, the results look even simpler. Due to the results of Hardy [29], with

$$\beta(s) = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-s},$$

in this case we have

$$\zeta_y^0(s) = \left(\frac{L}{2\pi}\right)^{2s} 4\zeta_R(s)\beta(s)$$

and the final answer takes the simple form

$$\zeta'(0) = \ln s_{11}(im) - \frac{L}{\pi}\zeta_R\left(\frac{1}{2}\right)\beta\left(\frac{1}{2}\right)\int_{-\infty}^{\infty} dx\, V(x).$$

$$+ \sum_{(n,j)\in\mathbb{Z}^2/\{0\}} \left\{\ln s_{11}\left(i\sqrt{m^2 + \left(\frac{2\pi}{L}\right)^2 [n^2 + j^2]}\right)\right.$$

$$\left. + \frac{L}{4\pi}\frac{1}{\sqrt{n^2 + j^2}}\int_{-\infty}^{\infty} dx\, V(x)\right\}. \tag{3.9}$$

3.2. Example of a sphere

For M a $(d-1)$-dimensional unit sphere the eigenvalues are known to be [23]

$$\lambda_l = l(l + d - 2) = \left(l + \frac{d-2}{2}\right)^2 - \frac{(d-2)^2}{4}, \qquad l \in \mathbb{N}_0.$$

In case a sphere of radius a is considered the eigenvalues simply scale like $1/a^2$. Therefore we will always assume $a = 1$ as $a \neq 1$ follows trivially. The degeneracy $\deg(l; d-1)$ for each eigenvalue is given by

$$\deg(l; d-1) = (2l + d - 2)\frac{(l+d-3)!}{l!(d-2)!}, \qquad l \in \mathbb{N}_0.$$

To keep the analysis as simple as possible, we assume $W(y) = (d-2)^2/4$, which corresponds to conformal coupling in $(d-1)$ dimensions. The eigenvalues then become a complete square,

$$\lambda_l = \left(l + \frac{d-2}{2}\right)^2.$$

Remarks about the case where $W(y)$ is an arbitrary constant are made in appendix C.

$d = 2$: This is identical to the torus case with $L = 2\pi$ and nothing more needs to be said.

$d = 3$: On the 2-sphere we have $\lambda_l = \left(l + \frac{1}{2}\right)^2$ with degeneracy $\deg(l; d-1) = 2l + 1$. Therefore we see

$$\zeta_y^0(s) = 2\zeta_H\left(2s - 1; \frac{1}{2}\right) = 2(2^{2s-1} - 1)\zeta_R(2s - 1). \tag{3.10}$$

Here $\zeta_H(s; b)$ denotes as usual the Hurwitz zeta function. The relevant properties of $\zeta_y^0(s)$ are

$$\text{Res}\,\zeta_y^0\left(\frac{1}{2}\right) = 0, \qquad \text{PP}\,\zeta_y^0\left(\frac{1}{2}\right) = 0.$$

This shows the final answer for equation (3.4) reads

$$\zeta'(0) = \sum_{l=0}^{\infty} \left\{(2l + 1)\ln s_{11}\left(i\sqrt{m^2 + \left(l + \frac{1}{2}\right)^2}\right) + \int_{-\infty}^{\infty} dx\, V(x)\right\}.$$

3.3. Zeta function construction with over-subtraction

In $d = 2$ and $d = 3$, for numerical convenience one might decide to add and subtract more terms in equation (3.3) than just the leading term from equation (3.2). Writing down the answer for the case with $\lambda^2 > 0$, having in mind the needed changes if $\lambda = 0$ occurs, we get for the general case subtracting the two terms given in (3.2)

$$
\zeta'(0) = \sum_\lambda \left\{ \ln s_{11}(\mathrm{i}\sqrt{m^2 + \lambda^2}) + \frac{1}{2|\lambda|} \int_{-\infty}^\infty \mathrm{d}x\, V(x) - \frac{1}{8|\lambda|^3} \int_{-\infty}^\infty \mathrm{d}x\, V(x)(V(x) + 2m^2) \right\}
$$
$$
+ \left(\mathrm{Res}\, \zeta_y^0 \left(\frac{1}{2} \right) \ln 2 - \frac{1}{2} PP \zeta_y^0 \left(\frac{1}{2} \right) \right) \int_{-\infty}^\infty \mathrm{d}x\, V(x) + \frac{1}{8} \left(PP\, \zeta_y^0 \left(\frac{3}{2} \right) \right.
$$
$$
- \left. \mathrm{Res}\, \zeta_y^0 \left(\frac{3}{2} \right) [-2 + \ln 4] \right) \int_{-\infty}^\infty \mathrm{d}x\, V(x)(V(x) + 2m^2). \tag{3.11}
$$

This result is also valid in $d = 4$. For particular cases like the torus or the sphere the final answer is easily found from known properties of Epstein-type zeta functions; some details are given in appendices Appendix B and Appendix C. Higher dimensions and more subtractions could be considered if necessary.

4. Conclusions

To conclude, we have presented new simplified explicit formulae for the one-loop vacuum energy when the underlying spacetime manifold is separable. The results have been derived using the zeta function method in conjunction with dimensional regularization and renormalization. The relation to dimensionally regularized Feynman diagrams is exactly as discussed in [16]. The cases considered here include (i) a static, radially symmetric background field in $(3 + 1)$- and $(2 + 1)$-dimensions, at both zero and non-zero temperatures; and (ii) a non-trivial domain-wall profile with a compact transverse manifold such as a sphere or a torus. The analysis is ultimately based on the Gel'fand–Yaglom theorem for the determinant of an ordinary differential operator, but extended to incorporate the necessary regularization and renormalization that appears in higher dimensions. Computationally, the final expressions are finite and convergent, and are considerably simpler than the corresponding expressions for example in [7, 8].

Acknowledgments

Appendix A. Asymptotic decomposition of $\zeta_{as}(s)$

In this appendix, we provide some details for the analysis of $\zeta_{as}(s)$ and of the finite temperature case for the situation of a spherically symmetric background field, see section 2.

Equations (2.7), (2.10)–(2.12) suggest to introduce the function

$$
f(s, c, b) = \frac{1}{\beta} \sum_{n=-\infty}^\infty \sum_{l=0}^\infty \deg(l; d-1) \nu^{-2s-c} \left(1 + \frac{(p_n^2 + m^2)r^2}{\nu^2} \right)^{-s-b}.
$$

We then find the representation

$$
\zeta_{as}(s) = -\frac{\Gamma\left(s+\frac{1}{2}\right)}{2\sqrt{\pi}\,\Gamma(s)} \int_0^\infty dr\, W(r) r^{1+2s} f\left(s, 1, \frac{1}{2}\right)
$$

$$
+ \frac{\Gamma\left(s+\frac{3}{2}\right)}{4\sqrt{\pi}\,\Gamma(s)} \int_0^\infty dr\, W(r) r^{1+2s} \left(W(r)r^2 - \frac{1}{2}\right) f\left(s, 3, \frac{3}{2}\right)
$$

$$
+ \frac{\Gamma\left(s+\frac{5}{2}\right)}{2\sqrt{\pi}\,\Gamma(s)} \int_0^\infty dr\, W(r) r^{1+2s} f\left(s, 3, \frac{5}{2}\right)
$$

$$
- \frac{\Gamma\left(s+\frac{7}{2}\right)}{6\sqrt{\pi}\,\Gamma(s)} \int_0^\infty dr\, W(r) r^{1+2s} f\left(s, 3, \frac{7}{2}\right). \tag{A.1}
$$

In order to evaluate $\zeta'_{as}(0)$ we mimic the process employed for $\zeta'_f(0)$. We explain the details for $f(s, 1, 1/2)$ as all other terms are obtained accordingly. First note that

$$
f\left(s, 1, \frac{1}{2}\right) = \frac{1}{\beta} \sum_{n=-\infty}^\infty \sum_{l=0}^\infty \deg(l; d-1)\left(v^2 + p_n^2 r^2\right)^{-s-\frac{1}{2}} \left[\left(1 + \frac{m^2 r^2}{v^2 + p_n^2 r^2}\right)^{-s-\frac{1}{2}}\right.
$$

$$
\left. - 1 + \left(s+\frac{1}{2}\right)\frac{m^2 r^2}{v^2 + p_n^2 r^2}\right]
$$

$$
+ \frac{1}{\beta} \sum_{n=-\infty}^\infty \sum_{l=0}^\infty \deg(l; d-1)\left(v^2 + p_n^2 r^2\right)^{-s-\frac{1}{2}} \left(1 - \left(s+\frac{1}{2}\right)\frac{m^2 r^2}{v^2 + p_n^2 r^2}\right).
$$

The first term is by construction well defined at $s = 0$, and for the second term an analytical continuation to $s = 0$ has to be performed. To this aim we introduce the Epstein-type zeta function

$$
E_d^{(k)}(s, a) = \frac{1}{\beta} \sum_{n=-\infty}^\infty \sum_{l=0}^\infty \deg(l; d-1)\frac{v^k}{(v^2 + a^2 n^2)^s}. \tag{A.2}
$$

We then have for the first term in (A.1)

$$
\zeta_{as,1}(s) = \zeta_{as,1}^{(1)}(s) + \zeta_{as,1}^{(2)}(s)
$$

with

$$
\zeta_{as,1}^{(1)}(s) = -\frac{1}{2\sqrt{\pi}}\frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(s)} \int_0^\infty dr\, W(r) r^{1+2s}
$$

$$
\times \left\{ E_d^{(0)}\left(s+\frac{1}{2}, \frac{2\pi r}{\beta}\right) - \left(s+\frac{1}{2}\right)m^2 r^2 E_d^{(0)}\left(s+\frac{3}{2}, \frac{2\pi r}{\beta}\right) \right\},
$$

$$
\zeta_{as,1}^{(2)}(s) = -\frac{1}{2} \int_0^\infty dr\, W(r) r \frac{1}{\beta} \sum_{n=-\infty}^\infty \sum_{l=0}^\infty \deg(l; d-1)\left(v^2 + p_n^2 r^2\right)^{-s-\frac{1}{2}}
$$

$$
\times \left[\left(1 + \frac{m^2 r^2}{v^2 + p_n^2 r^2}\right)^{-s-\frac{1}{2}} - 1 + \left(s+\frac{1}{2}\right)\frac{m^2 r^2}{v^2 + p_n^2 r^2}\right].
$$

Proceeding in the same fashion with the other terms in (A.1), equations (2.17) and (2.18) are established. Whereas the contribution $\zeta_{as}^{(2)\prime}(0)$ cancels with terms in $\zeta'_f(0)$, namely with the

14

last term in (2.13), the term $\zeta_{as}^{(1)}(s)$ contributes

$$
\begin{aligned}
\zeta_{as}^{(1)\prime}(0) = {} & -\frac{1}{2\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}s}\Bigg|_{s=0} \frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(s)} \int_0^\infty \mathrm{d}r\, W(r) r^{1+2s} \\
& \times \left(E_d^{(0)}\left(s+\frac{1}{2},\frac{2\pi r}{\beta}\right) - \left(s+\frac{1}{2}\right) m^2 r^2 E_d^{(0)}\left(s+\frac{3}{2},\frac{2\pi r}{\beta}\right) \right) \\
& + \frac{1}{4\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}s}\Bigg|_{s=0} \frac{\Gamma\left(s+\frac{3}{2}\right)}{\Gamma(s)} \int_0^\infty \mathrm{d}r\, W(r) r^{1+2s} \left(W(r) r^2 - \frac{1}{2}\right) E_d^{(0)}\left(s+\frac{3}{2},\frac{2\pi r}{\beta}\right) \\
& + \frac{1}{2\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}s}\Bigg|_{s=0} \frac{\Gamma\left(s+\frac{5}{2}\right)}{\Gamma(s)} \int_0^\infty \mathrm{d}r\, W(r) r^{1+2s} E_d^{(2)}\left(s+\frac{5}{2},\frac{2\pi r}{\beta}\right) \\
& - \frac{1}{6\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}s}\Bigg|_{s=0} \frac{\Gamma\left(s+\frac{7}{2}\right)}{\Gamma(s)} \int_0^\infty \mathrm{d}r\, W(r) r^{1+2s} E_d^{(4)}\left(s+\frac{7}{2},\frac{2\pi r}{\beta}\right).
\end{aligned}
$$

To find an explicit answer for $\zeta_{as}^{(1)\prime}(0)$ we are left to do the analysis of the zeta function $E_d^{(k)}(s,a)$ in equation (A.2). To perform the limit $\beta \to \infty$ the best way to proceed is to perform a Poisson resummation in the n-summation [1, 18, 36]. This leads to the following result:

$$
\begin{aligned}
E_d^{(k)}(s,a) = {} & \frac{\sqrt{\pi}}{a\beta} \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \sum_{l=0}^\infty \deg(l;d-1) \nu^{k+1-2s} \\
& + \frac{4\pi^s}{\beta \Gamma(s) a^{s+\frac{1}{2}}} \sum_{l=0}^\infty \deg(l;d-1) \nu^{k+\frac{1}{2}-s} \sum_{n=1}^\infty n^{s-\frac{1}{2}} K_{\frac{1}{2}-s}\left(2\pi\nu\frac{n}{a}\right).
\end{aligned} \tag{A.3}
$$

The second line is analytic for all values of s, the first line contains first-order singularities at certain s-values that need to be considered in detail. Using the Laurent series of meromorphic functions with a first-order pole at $s=s_0$ we expand

$$
E_d^{(k)}(s,a) = \frac{1}{s-s_0} \operatorname{Res} E_d^{(k)}(s_0,a) + \operatorname{PP} E_d^{(k)}(s_0,a) + \mathcal{O}(s-s_0),
$$

to obtain $\zeta_{as}^{(1)\prime}(0)$ in the form

$$
\begin{aligned}
\zeta_{as}^{(1)\prime}(0) = {} & -\frac{1}{16} \int_0^\infty \mathrm{d}r\, r W(r) \Bigg[8\left(\operatorname{PP} E_d^{(0)}\left(\frac{1}{2},\frac{2\pi r}{\beta}\right) + \operatorname{Res} E_d^{(0)}\left(\frac{1}{2},\frac{2\pi r}{\beta}\right) \ln\left(\frac{r^2}{4}\right) \right) \\
& + (1-2r^2[W(r)+2m^2]) \left(\operatorname{PP} E_d^{(0)}\left(\frac{3}{2},\frac{2\pi r}{\beta}\right) \right. \\
& \left. + \operatorname{Res} E_d^{(0)}\left(\frac{3}{2},\frac{2\pi r}{\beta}\right) \left[\ln\left(\frac{r^2}{4}\right) + 2 \right] \right) \\
& - 6\left(\operatorname{PP} E_d^{(2)}\left(\frac{5}{2},\frac{2\pi r}{\beta}\right) + \operatorname{Res} E_d^{(2)}\left(\frac{5}{2},\frac{2\pi r}{\beta}\right) \left[\ln\left(\frac{r^2}{4}\right) + \frac{8}{3} \right] \right) \\
& + 5\left(\operatorname{PP} E_d^{(4)}\left(\frac{7}{2},\frac{2\pi r}{\beta}\right) + \operatorname{Res} E_d^{(4)}\left(\frac{7}{2},\frac{2\pi r}{\beta}\right) \left[\ln\left(\frac{r^2}{4}\right) + \frac{46}{15} \right] \right) \Bigg]. \tag{A.4}
\end{aligned}
$$

The relevant properties of $E_d^{(k)}(s,2\pi r/\beta)$ are easily found from (A.3). First note that the series over the Bessel functions $K_{1/2-s}(\nu n\beta/r)$ vanishes exponentially fast as $\beta \to \infty$. Its contribution at finite temperature is obtained by simply substituting the s-values needed in (A.4). In some detail, if we define

$$
C_d^{(k)}(s,a) = \frac{4\pi^s}{\beta \Gamma(s) a^{s+\frac{1}{2}}} \sum_{l=0}^\infty \deg(l;d-1) \nu^{k+\frac{1}{2}-s} \sum_{n=1}^\infty n^{s-\frac{1}{2}} K_{\frac{1}{2}-s}\left(2\pi\nu\frac{n}{a}\right),
$$

then its contribution follows from replacing PP $E_d^{(k)}(s, 2\pi r/\beta)$ by $C_d^{(k)}(s, 2\pi r\beta)$ in (A.4); note there are no contributions to the residue terms in (A.4). The contributions so obtained have to be added to the zero temperature result (2.24) following from the first line in (A.3).

So in the following, let us concentrate on the contributions from the first line in (A.3), which are the $T = 0$ contributions. In $d = 4$ we have $\deg(l; d-1) = (2l+1)$ and $\nu = l + 1/2$ which shows

$$E_4^{(k)}\left(s, \frac{2\pi r}{\beta}\right) = \frac{1}{\sqrt{\pi}r} \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \zeta_H\left(2s - k - 2; \frac{1}{2}\right). \tag{A.5}$$

The relevant expansions are

$$E_4^{(0)}\left(s, \frac{2\pi r}{\beta}\right) = \frac{1}{24\pi r} \frac{1}{s - \frac{1}{2}} - \frac{1}{\pi r}\zeta_R'(-1) + \mathcal{O}\left(s - \frac{1}{2}\right),$$

$$E_4^{(0)}\left(s, \frac{2\pi r}{\beta}\right) = \frac{1}{\pi s} \frac{1}{s - \frac{3}{2}} + \frac{2}{\pi r}(-1 + \gamma + \ln 8) + \mathcal{O}\left(s - \frac{3}{2}\right),$$

$$E_4^{(2)}\left(s, \frac{2\pi r}{\beta}\right) = \frac{2}{3\pi r} \frac{1}{s - \frac{5}{2}} + \frac{2}{9\pi r}(-5 + 6\gamma + 18\ln 2) + \mathcal{O}\left(s - \frac{5}{2}\right),$$

$$E_4^{(4)}\left(s, \frac{2\pi r}{\beta}\right) = \frac{8}{15\pi r} \frac{1}{s - \frac{7}{2}} + \frac{4}{15\pi r}\left(-\frac{47}{15} + 4\gamma + 12\ln 2\right) + \mathcal{O}\left(s - \frac{7}{2}\right).$$

Using these in (A.4) the result is equation (2.23), namely

$$\zeta_{as}^{(1)\prime}(0) = \frac{1}{24\pi} \int_0^\infty dr\, W(r)(12\zeta_R'(-1) + \gamma + 3\ln 2 + 6r^2(W(r) + 2m^2)(\gamma + \ln(4mr))).$$

As mentioned before, in $d = 3$, equation (2.3) can only be used for $l \geqslant 1$. The contributions from those modes are still given by (A.4) with the sum in (A.3) starting at $l = 1$. For $l \geqslant 1$, the degeneracy $\deg(l; 2) = 2$ and (A.3) is replaced by

$$E_3^{(k)}(s, a) = \frac{2\sqrt{\pi}}{a\beta} \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \zeta_R(2s - k - 1)$$

$$+ \frac{8\pi^s}{\beta\Gamma(s)a^{s+\frac{1}{2}}} \sum_{l=1}^\infty \sum_{n=1}^\infty l^{k+\frac{1}{2}-s} n^{s-\frac{1}{2}} K_{\frac{1}{2}-s}\left(\frac{2\pi ln}{a}\right). \tag{A.6}$$

For reasons explained, concentrating on $T = 0$, the relevant expansions are

$$E_3^{(0)}\left(s, \frac{2\pi r}{\beta}\right) = -\frac{1}{2\pi r} \frac{1}{s - \frac{1}{2}} - \frac{1}{\pi r}\ln(4\pi) + \mathcal{O}\left(s - \frac{1}{2}\right),$$

$$E_3^{(0)}\left(s, \frac{2\pi r}{\beta}\right) = \frac{\pi}{3r} + \mathcal{O}\left(s - \frac{3}{2}\right),$$

$$E_3^{(2)}\left(s, \frac{2\pi r}{\beta}\right) = \frac{2\pi}{9r} + \mathcal{O}\left(s - \frac{5}{2}\right),$$

$$E_3^{(4)}\left(s, \frac{2\pi r}{\beta}\right) = \frac{8\pi}{45r} + \mathcal{O}\left(s - \frac{7}{2}\right).$$

The $l = 0$ contribution is

$$\zeta_0(s) = \frac{\sin \pi s}{\pi\beta} \sum_{n=-\infty}^\infty \int_{\sqrt{p_n^2 + m^2}}^\infty dk\left(k^2 - p_n^2 - m^2\right)^{-s} \frac{d}{dk} \ln f_0(ik).$$

Adding and subtracting the large-k behavior of $\ln f_0(\mathrm{i}k)$ this is rewritten along the lines presented as

$$\zeta_0(s) = \frac{\sin \pi s}{\pi \beta} \sum_{n=-\infty}^{\infty} \int_{\sqrt{p_n^2+m^2}}^{\infty} \mathrm{d}k \left(k^2 - p_n^2 - m^2\right)^{-s} \frac{\mathrm{d}}{\mathrm{d}k} \left[\ln f_0(\mathrm{i}k) - \frac{1}{2k} \int_0^{\infty} \mathrm{d}r\, W(r)\right]$$
$$- \frac{1}{2\sqrt{\pi}} \frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(s)} Z\left(s+\frac{1}{2}, \left(\frac{2\pi}{\beta}\right)^2 \Big| m^2\right) \int_0^{\infty} \mathrm{d}r\, W(r),$$

where [1, 18, 36]

$$Z(s, a^2|m^2) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} (a^2 n^2 + m^2)^{-s}$$
$$= \frac{\sqrt{\pi}}{a\beta} \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} m^{1-2s} + \frac{4\pi^s m^{\frac{1}{2}-s}}{\beta \Gamma(s) a} \sum_{n=1}^{\infty} \left(\frac{n}{a}\right)^{s-\frac{1}{2}} K_{\frac{1}{2}-s}\left(2\pi m \frac{n}{a}\right). \tag{A.7}$$

At $T = 0$ this gives for the determinant of the $l = 0$ contribution

$$\zeta_0'(0) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}p \left[\ln f_0(\mathrm{i}\sqrt{p^2+m^2}) - \frac{1}{2\sqrt{p^2+m^2}} \int_0^{\infty} \mathrm{d}r\, W(r)\right]$$
$$+ \frac{1}{2\pi} \ln m \int_0^{\infty} \mathrm{d}r\, W(r).$$

At finite temperature a series over Bessel functions resulting in (A.7) has to be added as described below (A.4).

Adding up the $l = 0$ and $l \geqslant 1$ contributions, we obtain

$$\zeta'(0) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}p \left[\ln f_0(\mathrm{i}\sqrt{p^2+m^2}) - \frac{1}{2\sqrt{p^2+m^2}} \int_0^{\infty} \mathrm{d}r\, W(r)\right]$$
$$- \frac{1}{\pi} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}p \left[\ln f_l(\mathrm{i}\sqrt{p^2+m^2}) - \frac{1}{2} \int_0^{\infty} \mathrm{d}r\, \frac{r W(r)}{(l^2 + p^2 r^2)^{\frac{1}{2}}}\right.$$
$$+ \frac{1}{8} \int_0^{\infty} \mathrm{d}r\, \frac{r^3 W(r)(W(r) + 2m^2)}{(l^2 + p^2 r^2)^{\frac{3}{2}}}$$
$$\left. - \frac{1}{16} \int_0^{\infty} \mathrm{d}r\, \frac{r W(r)}{(l^2 + p^2 r^2)^{\frac{3}{2}}} \left(1 - \frac{6l^2}{l^2 + p^2 r^2} + \frac{5l^4}{(l^2 + p^2 r^2)^2}\right)\right]$$
$$+ \frac{1}{16\pi} \int_0^{\infty} \mathrm{d}r\, W(r) \left[\frac{\pi^2}{9}(1 + 6r^2(W(r) + 2m^2)) + 8\ln(2\pi m r)\right].$$

This final answer involves oversubtractions we made, which, despite the fact that it makes a numerical evaluation easier has the disadvantage of looking more complicated. Subtracting what is strictly necessary for the procedure in $d = 3$ we obtain

$$\zeta'(0) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}p \left[\ln f_0(\mathrm{i}\sqrt{p^2+m^2}) - \frac{1}{2\sqrt{p^2+m^2}} \int_0^{\infty} \mathrm{d}r\, W(r)\right]$$
$$- \frac{1}{\pi} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}p \left[\ln f_l(\mathrm{i}\sqrt{p^2+m^2}) - \frac{1}{2} \int_0^{\infty} \mathrm{d}r\, \frac{r W(r)}{(l^2 + p^2 r^2)^{\frac{1}{2}}}\right]$$
$$+ \frac{1}{2\pi} \int_0^{\infty} \mathrm{d}r\, W(r) \ln(2\pi r m).$$

Appendix B. Zeta function properties on the torus

In this appendix, we collect the information needed to evaluate the expression for $\zeta'(0)$ given in equation (3.11) for the example of a torus.

To exploit equation (3.11) for the torus, in $d = 2$ the only additional information needed beyond equation (3.6) is $\zeta_y^0(3/2)$. From equation (3.5)

$$\text{Res}\,\zeta_y^0\left(\frac{3}{2}\right) = 0 \qquad \text{and} \qquad \zeta_y^0\left(\frac{3}{2}\right) = \frac{L^3}{4\pi^3}\zeta_R(3).$$

In $d = 3$ we use equation (3.7) to see $\text{Res}\,\zeta_y^0(3/2) = 0$ and

$$\zeta_y^0\left(\frac{3}{2}\right) = Z_2\left(\frac{3}{2}; \left(\frac{2\pi}{L_1}\right)^2, \left(\frac{2\pi}{L_2}\right)^2\right)$$

$$= \frac{L_2^3}{4\pi^3}\zeta_R(3) + \frac{L_1^2 L_2}{12\pi} + \frac{2L_1 L_2^2}{\pi^2}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{n}{j}K_1\left(2\pi n j\frac{L_2}{L_1}\right).$$

For $d = 4$ we introduce

$$Z_3(s; w_1, w_2, w_3) = \sum_{(n_1, n_2, n_3)\in\mathbb{Z}^3\setminus\{0\}}\left(w_1 n_1^2 + w_2 n_2^2 + w_3 n_3^2\right)^{-s}$$

and have

$$\zeta_y^0(s) = Z_3\left(s; \left(\frac{2\pi}{L_1}\right)^2, \left(\frac{2\pi}{L_2}\right)^2, \left(\frac{2\pi}{L_3}\right)^2\right).$$

The analytical continuation of $Z_3(s; w_1, w_2, w_3)$ to a meromorphic function in the complex plane reads [1, 18, 36]

$$Z_3(s; w_1, w_2, w_3) = \frac{2}{w_3^s}\zeta_R(2s) + \sqrt{\frac{\pi}{w_3}}\frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)}Z_2\left(s - \frac{1}{2}; w_1, w_2\right)$$

$$+ \frac{4\pi^s}{\Gamma(s)\sqrt{w_3}}\sum_{n=1}^{\infty}\sum_{(n_1, n_2)\in\mathbb{Z}^2\setminus\{0\}}\left(\frac{\left[w_1 n_1^2 + w_2 n_2^2\right]w_3}{n^2}\right)^{\frac{1}{2}\left(\frac{1}{2} - s\right)}$$

$$\times K_{\frac{1}{2}-s}\left(\frac{2\pi n}{\sqrt{w_3}}\sqrt{w_1 n_1^2 + w_2 n_2^2}\right), \tag{B.1}$$

which we need to analyze further about the points $s = 1/2$ and $s = 3/2$. As intermediate results we note that

$$\text{Res}\,Z_3\left(\frac{1}{2}; w_1, w_2, w_3\right) = \frac{1}{\sqrt{w_3}} + \frac{Z_2(0; w_1, w_2)}{\sqrt{w_3}},$$

$$\text{PP}\,Z_3\left(\frac{1}{2}; w_1, w_2, w_3\right) = \frac{2\gamma - \ln w_3}{\sqrt{w_3}} + \frac{1}{\sqrt{w_3}}[Z_2'(0; w_1, w_2) + Z_2(0; w_1, w_2)\ln 4]$$

$$+ \frac{4}{\sqrt{w_3}}\sum_{n=1}^{\infty}\sum_{(n_1, n_2)\in\mathbb{Z}^2\setminus\{0\}}K_0\left(\frac{2\pi n}{\sqrt{w_3}}\sqrt{w_1 n_1^2 + w_2 n_2^2}\right),$$

$$\text{Res}\,Z_3\left(\frac{3}{2}; w_1, w_2, w_3\right) = \frac{2}{\sqrt{w_3}}\text{Res}\,Z_2(1; w_1, w_2),$$

18

$$\text{PP} \, Z_3 \left(\frac{3}{2}; w_1, w_2, w_3 \right) = \frac{2}{w_3^{\frac{3}{2}}} \zeta_R(3) + \frac{2}{\sqrt{w_3}} [\text{PP} \, Z_2(1; w_1, w_2) + \text{Res} \, Z_2(1; w_1, w_2)(\ln 4 - 2)]$$

$$+ \frac{8\pi}{w_3} \sum_{n=1}^{\infty} \sum_{(n_1, n_2) \in \mathbb{Z}^2/\{0\}} \frac{n}{\sqrt{w_1 n_1^2 + w_2 n_2^2}} K_1 \left(\frac{2\pi n}{\sqrt{w_3}} \sqrt{w_1 n_1^2 + w_2 n_2^2} \right).$$

equation (3.7) is then used to find the relevant properties of $Z_2(s; w_1, w_2)$ about $s = 0$ and $s = 1$, namely

$$Z_2(0; w_1, w_2) = -1,$$

$$Z_2'(0; w_1, w_2) = \ln \left(\frac{w_2}{4\pi^2} \right) + \frac{\pi}{3} \sqrt{\frac{w_1}{w_2}} - 4 \sum_{n=1}^{\infty} \ln \left(1 - e^{-2\pi n \sqrt{\frac{w_1}{w_2}}} \right),$$

$$\text{Res} \, Z_2(1; w_1, w_2) = \frac{\pi}{\sqrt{w_1 w_2}},$$

$$\text{PP} \, Z_2(1; w_1, w_2) = \frac{\pi^2}{3 w_2} + \frac{\pi}{\sqrt{w_1 w_2}} (2\gamma - \ln(4w_1)) - \frac{4\pi}{(w_1 w_2)^{\frac{3}{4}}} \sum_{n=1}^{\infty} \ln \left(1 - e^{-2\pi n \sqrt{\frac{w_1}{w_2}}} \right).$$

These results are used to produce the final answer for equation (3.11) in $d = 4$ by substituting

$$\text{Res} \, Z_3 \left(\frac{1}{2}; \left(\frac{2\pi}{L_1} \right)^2, \left(\frac{2\pi}{L_2} \right)^2, \left(\frac{2\pi}{L_3} \right)^2 \right) = 0,$$

$$\text{PP} \, Z_3 \left(\frac{1}{2}; \left(\frac{2\pi}{L_1} \right)^2, \left(\frac{2\pi}{L_2} \right)^2, \left(\frac{2\pi}{L_3} \right)^2 \right)$$

$$= \frac{L_3}{2\pi} \left[2\gamma + \ln \left(\frac{L_3}{16\pi^2 L_2} \right) + \frac{\pi}{3} \frac{L_2}{L_1} - 4 \sum_{n=1}^{\infty} \ln \left(1 - e^{-2\pi n \frac{L_2}{L_1}} \right) \right]$$

$$+ \frac{2L_3}{\pi} \sum_{n=1}^{\infty} \sum_{(n_1, n_2) \in \mathbb{Z}^2/\{0\}} K_0 \left(2\pi n L_3 \sqrt{\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2}} \right),$$

$$\text{Res} \, Z_3 \left(\frac{3}{2}; \left(\frac{2\pi}{L_1} \right)^2, \left(\frac{2\pi}{L_2} \right)^2, \left(\frac{2\pi}{L_3} \right)^2 \right) = \frac{1}{4\pi^2} L_1 L_2 L_3,$$

$$\text{PP} \, Z_3 \left(\frac{3}{2}; \left(\frac{2\pi}{L_1} \right)^2, \left(\frac{2\pi}{L_2} \right)^2, \left(\frac{2\pi}{L_3} \right)^2 \right) = \frac{1}{4\pi^3} L_3^3 \zeta_R(3)$$

$$+ \frac{L_3}{\pi} \left[\frac{L_2^2}{12} + \frac{L_1 L_2}{4\pi} \left(2\gamma - 2 - 2 \ln \left(\frac{2\pi}{L_1} \right) \right) - \frac{(L_1 L_2)^{\frac{3}{2}}}{2\pi^2} \sum_{n=1}^{\infty} \ln \left(1 - e^{-2\pi n \frac{L_2}{L_1}} \right) \right]$$

$$+ \frac{L_3^2}{\pi^2} \sum_{n=1}^{\infty} \sum_{(n_1, n_2) \in \mathbb{Z}^2/\{0\}} \frac{n}{\sqrt{\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2}}} K_1 \left(2\pi n L_3 \sqrt{\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2}} \right).$$

Higher dimensions could be considered as well by using the generalization of equation (B.1) to higher dimensions [1, 18, 36].

Appendix C. Zeta function properties on the sphere

Let us now turn our attention to the analysis of equation (3.11) for the case of the sphere. In $d = 2$ this is the same as the torus and pertinent results can be found above.

$d = 3$: One easily obtains from equation (3.10) that

$$\text{Res}\,\zeta_y^0\left(\frac{3}{2}\right) = 0, \qquad \text{PP}\,\zeta_y^0\left(\frac{3}{2}\right) = \pi^2 \tag{C.1}$$

and from here equation (3.11) can be written down immediately for this example.

$d = 4$: Here the eigenvalues are $\lambda_l = (l + 1)^2$ with degeneracy $\deg(l; 3) = (l + 1)^2$ and so the relevant zeta function is

$$\zeta_y^0(s) = \zeta_R(2s - 2).$$

Therefore everything needed is available, in particular

$$\text{Res}\,\zeta_y^0\left(\frac{1}{2}\right) = 0, \qquad \text{PP}\,\zeta_y^0\left(\frac{1}{2}\right) = -\frac{1}{12}, \qquad \text{Res}\,\zeta_y^0\left(\frac{3}{2}\right) = \frac{1}{2}, \qquad \text{PP}\,\zeta_y^0\left(\frac{3}{2}\right) = \gamma,$$

and again the final answer for $\zeta'(0)$ in equation (3.11) is trivially written down. Also higher dimensions could be considered along the same lines.

For the more general case of $W(y) = c + (d - 2)^2/4$, c an arbitrary constant, the eigenvalues have the form

$$\lambda_l = \left(l + \frac{d - 2}{2}\right)^2 + c.$$

An analytical continuation of the zeta function $\zeta_y^0(s)$ in terms of Hurwitz zeta functions is easily obtained with the help of a binomial expansion in powers of c. This is a well-known procedure and we do not present more details; see, e.g., [13, 19].

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